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Self-similar Delone sets and quasicrystals

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Abstract. In this paper we answer the question, whether any Delone set $\Lambda \subset \mathbb{R}^n$, invariant under quasiaddition of Berman and Moody, can be identified with a cut and project quasicrystal. For any such set Λ, we find an acceptance window Ω, which is bounded but has only convex interior. The cut and project quasicrystal $\Sigma(\Omega)$ is then identified with an affine image of Λ. Constructive methods used in the paper, allow one, in principle, to put bounds on Ω from a given fragment of a Delone set.

1. Introduction

This paper studies properties of aperiodic deterministic point sets with the Delone (or Delaunay) property, called quasicrystalline sets or simply quasicrystals in $\mathbb{R}^n$. The sets $\Sigma(\Omega)$ considered here are of the 'cut and project' type with bounded convex 'acceptance window' Ω. Relative to a suitable basis of $\mathbb{R}^n$, the coordinates of their points are in the ring of integers $\mathbb{Z}[\tau]$ of the field $\mathbb{Q}[\tau]$, where $\mathbb{Q}[\tau]$ is the algebraic extension of rational numbers by the golden mean $\tau = \frac{1}{2}(1+\sqrt{5})$.

In analogy with conventional addition on crystalline point sets, one searched for a binary operation under which a certain class of aperiodic sets would be invariant. Such an operation, called quasiaddition or equivalently τ-inflation, was introduced by Berman and Moody [2]. They have shown that cut and project quasicrystals with bounded convex acceptance windows are invariant under this operation. Until then there was no analogue of the ordinary addition of lattice points on quasicrystals.

The boundedness and convexity of Ω assure respectively the Delone property and τ-inflation invariance of $\Sigma(\Omega)$ [9]. The purpose of this paper is to investigate the opposite implication: when a Delone set closed under quasiaddition can be identified with a cut and project quasicrystal? In the main theorem (theorem 2.4) of this paper, we find, for each such set $\Lambda \subset \mathbb{R}^n$, an acceptance window Ω, which is bounded with convex interior. It is the boundary of Ω that complicates an exhaustive answer. The cut and project quasicrystal $\Sigma(\Omega)$ is then an affine image of Λ.

This paper can be considered as a direct answer to the question asked in [4, (question (iii) p 149)]. It was natural to anticipate the main part of our result, namely that concerning boundedness and convexity of the interior of the corresponding acceptance window.

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Nevertheless, any conclusion is reliable only after it has been proven. Moreover our problem, the opposite implication to that of Berman and Moody which we study here, is far from being just a scientific curiosity. On it hinges a way in which one can deduce information about the size and shape of the acceptance window when a fragment (i.e. finite size) quasicrystal is given, for example by an experiment. Naturally the acceptance window cannot be determided precisely, only bounds of its size can be found [7].

Cut and project quasicrystals can be defined for other algebraic irrationalities than τ [3, 8], together with the corresponding quasiaddition [1].

By definition (2), the cut and project quasicrystals are subsets of a $\mathbb{Z}[\tau]$-lattice in $\mathbb{R}^n$ determined by their acceptance window. Therefore our first task is to find a mapping of an arbitrary given Delone τ-inflation invariant set in $\mathbb{R}^n$ into a $\mathbb{Z}[\tau]$-lattice in a way, which would preserve the Delone property and the τ-inflation invariance. Then the second task is to determine an acceptance window which would allow one to identify the mapped set as the corresponding quasicrystal.

Since every cut and project quasicrystal with convex Ω is τ-inflation invariant, one may expect the opposite implication to hold as well. It turns out that not every Delone τ-inflation invariant set is transformable into a cut and project quasicrystal with convex Ω. Instead, the implication leads to Ω with convex interior only. Its boundary is subject to some additional conditions. Our definition of 'quasiconvexity' includes all the requirements one has to impose on Ω.

The main theorem is stated in section 2. The rest of the paper essentially contains its proof. However, some parts of the demonstration are of independent interest. The proof of the theorem in one dimension is found in section 3. It is frequently called for in the general case. In section 4 we have brought together some auxiliary statements which were indispensable for the proof. The theorem is then proven in section 5. The quasiconvexity notion is introduced in definition 5.4. It then allows one to find the necessary and sufficient condition for a cut and project quasicrystal to be closed under τ-inflation. The last section consists of an example.

2. Definitions and the main theorem

First we introduce notation and definitions which are needed to formulate the theorem summarizing the main result of the paper, and we also recall some properties of defined objects which are subsequently used. Let us single out the Delone property of a point set, and the operations of τ- and τ'-inflations, where $\tau = \frac{1}{2}(1+\sqrt{5})$ and $\tau' = \frac{1}{2}(1-\sqrt{5})$ are the solutions of the algebraic equation $x^2 = x + 1$.

In $\mathbb{Q}[\tau] = \{s + t\tau \mid s, t \in \mathbb{Q}\}$, the extension of the rational numbers by τ, there is an automorphism $': \mathbb{Q}[\tau] \to \mathbb{Q}[\tau]$ given by $\tau \to \tau'$. The ring $\mathbb{Z}[\tau]$ of integers of $\mathbb{Q}[\tau]$ is the set

$$\mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}.$$

Let us recall the following about $\mathbb{Z}[\tau]$.

- It is dense in the set of all real numbers $\mathbb{R}$.
- It is a ring of principal ideals, i.e. all ideals in $\mathbb{Z}[\tau]$ are of the form $\xi\mathbb{Z}[\tau]$ for some $\xi \in \mathbb{Z}[\tau]$. All sets of this form for $\xi \neq 0$ are dense in $\mathbb{R}$.
- In particular, $\mathbb{Z}[\tau]$ is a unique factorization domain, hence the greatest common divisor is well defined up to multiplication by a divisor of 1. If $d = \gcd\{F\}$ is the greatest common divisor of a set $F \subset \mathbb{Z}[\tau]$, then d' is the greatest common divisor of the set F' of conjugated points to points of F.

• The group of units of $\mathbb{Z}[\tau]$ consists of $\{\pm\tau^k \mid k \in \mathbb{Z}\}$. (An element $y \in \mathbb{Z}[\tau]$ is a unit iff $yy' = \pm 1$.) In a ring R of principal ideals the following statement holds. For $u, v \in R$, $\gcd\{u, v\} = 1$, the equation $ux + vy = 1$ has a solution in R.

An n-dimensional $\mathbb{Z}[\tau]$-lattice $M \subset \mathbb{R}^n$ is a free $\mathbb{Z}[\tau]$-module of the rank n, $M = \sum_i \mathbb{Z}[\tau]\alpha_i$, with α_i a basis of $\mathbb{R}^n$. We consider the $\mathbb{Z}[\tau]$-lattices for which the standard scalar product takes values in $\mathbb{Q}[\tau]$. Let us recall some facts following from general properties of modules. Two bases of a $\mathbb{Z}[\tau]$-module M are related by a matrix A of determinant $\det A = \pm\tau^k$ for some $k \in \mathbb{Z}$. A set $L \subset M$ is a submodule of M, if it is a $\mathbb{Z}[\tau]$-module on itself. If L is a submodule of a module M, spanning the same $\mathbb{R}^n$, then there exist bases $f_1, \ldots, f_n$ and $e_1, \ldots, e_n$ of L and M respectively, such that $f_i = a_i e_i$, where $a_i \in \mathbb{Z}[\tau]$, $i = 1, \ldots, n$. Let $S \subset \mathbb{R}^n$. We denote the set

$$[S]^\tau = \left\{ u_1 + \sum_{i=1}^{m} \xi_i (u_i - u_1) \,\middle|\, \xi_i \in \mathbb{Z}[\tau], u_i \in S \right\}. \tag{1}$$

Clearly, this definition does not depend on the choice of u_1 in S. In the case that $0 \in S$, the set $[S]^\tau$ coincides with the $\mathbb{Z}[\tau]$-module generated by elements of S.

Let $*$ be a mapping $* : M \to \mathbb{R}^n$. It is called the 'star map' if it is semilinear with respect to the automorphism $'$, i.e. for any $x, y \in M$, and any $r \in \mathbb{Z}[\tau]$, one has $(x + ry)^* = x^* + r'y^*$, and M^* spans $\mathbb{R}^n$ over real numbers. On any $\mathbb{Z}[\tau]$-lattice M, one can define a star map by putting $\alpha_i^* = \alpha_i$ for all basis vectors.

Note that M^* is also a $\mathbb{Z}[\tau]$-lattice. If α_i is the basis of M, than α_i^* is the basis of M^*. Having an $S \subset M$, then

$$([S]^\tau)^* = [S^*]^\tau.$$

In general, $M^* \neq M$, but it may happen that M and M^* coincide even if $\alpha_i^* \neq \alpha_i$.

The point sets considered in this article are required to be Delone sets or, equivalently, to have the Delone property.

Definition 2.1. A set $\Lambda \subset \mathbb{R}^n$ is *Delone* if there exist $r_1, r_2 > 0$ such that:
(i) Λ is *uniformly discrete*: $(\forall x \in \Lambda)(B(x, r_1) \cap \Lambda = \{x\})$,
(ii) Λ is *relatively dense*: $(\forall\, x \in \mathbb{R}^n)(B(x, r_2) \cap \Lambda \neq \emptyset)$,
where $B(x, r) \subset \mathbb{R}^n$ is the n-dimensional ball of the radius r, centred at $x \in \mathbb{R}^n$.

Definition 2.2. Let M be a $\mathbb{Z}[\tau]$-lattice in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$. Let $\Sigma(\Omega)$ be the set of points

$$\Sigma(\Omega) = \{x \in M | x^* \in \Omega\}. \tag{2}$$

For Ω bounded with nonempty interior the set $\Sigma(\Omega)$ is called a quasicrystal. Ω is called an acceptance window for $\Sigma(\Omega)$.

According to [9], the boundedness of Ω, together with the fact that the interior Ω° is nonempty, assure the Delone property of $\Sigma(\Omega)$. For Ω bounded, (2) is the usual way to define a quasicrystal [10, 4] of cut and project type. Note that the stage for the quasicrystal is the $\mathbb{Z}[\tau]$-lattice M. Its star map image is the set $\Omega \cap M^*$. In the one-dimensional case one considers $M = \mathbb{Z}[\tau]$, and the star map reduces to $x^* = x'$, for $x \in \mathbb{Z}[\tau]$. In this case $(x')' = x$ and $(\mathbb{Z}[\tau])' = \mathbb{Z}[\tau]$. An example of a one-dimensional quasicrystal is found in (6).

Next recall the operation $\vdash$ on $\mathbb{R}^n$, called either *quasiaddition* or τ*-inflation* [2], and let us simultaneously introduce the analogous operation τ'-inflation denoted $\dashv$,

$$x \vdash y := \tau^2 x - \tau y \tag{3}$$

$$x \dashv y := (\tau')^2 x - \tau' y = \frac{x}{\tau^2} + \frac{y}{\tau}. \tag{4}$$

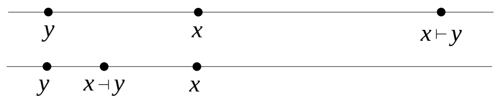

Figure 1. Geometrical meaning of definitions (3) and (4) of τ- and τ'-inflations.

Note that the τ'-inflation is a convex linear combination of x and y. Indeed, one has $0 < \tau^{-1} < 1$ and for any power of τ, the relation $\tau^{j+2} = \tau^{j+1} + \tau^j$ is valid.

Relation (3) is not a unique way to define the quasiaddition. There is an infinite series of similar operation one introduces in its place. For fixed x an any y, we have from (3)

$$x \vdash y = x - \tau(y - x). \tag{5}$$

The geometric content of (5) is visible in figure 1. It is an inflation of the distance $|y - x|$ at the point x by the factor $-\tau$. One could consider other scaling factors provided they are in $\mathbb{Z}[\tau]$ and the operation analogous to (4) is also convex. All such scaling factors form a one-dimensional quasicrystal

$$\Sigma\,([0, 1]) := \{x \in \mathbb{Z}[\tau] \mid 0 \leqslant x' \leqslant 1\}. \tag{6}$$

In [5] it was shown that $\Sigma(\Omega)$ with convex Ω are s-inflation invariant for $s \in \Sigma([0, 1])$. Therefore this study could be carried out using an s-inflation with $s \in \Sigma\,([0, 1])$ instead of only $s = -\tau$. Subsequently we only consider the $(-\tau)$-inflation, calling it traditionally 'τ-inflation'. Note that $-\tau$ is the lowest element of $\Sigma\,([0, 1])$ in absolute value, providing a nontrivial scaling. It appears unlikely that for general $s \in \Sigma([0, 1])$, one could obtain similar results as those presented in this paper.

Let us introduce the τ- and τ'-inflation closures, $A^{\vdash}$ and $A^{\dashv}$, for a set $A \subset \mathbb{R}^n$, as the minimal sets invariant under τ- and τ'-inflations respectively, containing A.

Subsequently, when working with elements of M, it is often advantageous to transfer the consideration to the M^*-side in order to be confined to a finite region. The one-to-one correspondence between the two can be written as

$$w \in \{u, v\}^{\vdash} \Longleftrightarrow w^* \in \{u^*, v^*\}^{\dashv} \qquad \forall u, v \in M. \tag{7}$$

Let us point out some properties of τ-inflation [2], which are used below. Similar properties involving τ' are obtained replacing $\vdash$ by $\dashv$. For any $x, y \in \mathbb{R}^n$:

(i) $x \vdash x = x$,

(ii) $x \vdash y = y \vdash (y \vdash x)$,

(iii) for any affine mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$, one has $\phi(x \vdash y) = \phi(x) \vdash \phi(y)$. In particular, for any $u \in \mathbb{R}^n$, $(x + u) \vdash (y + u) = (x \vdash y) + u$.

Later, while studying one-dimensional sets, we find useful the following lemma. It implies that for any set $S \subset \mathbb{Z}[\tau]$ of generators, the inflation closure $S^{\vdash}$ (or $S^{\dashv}$ respectively) is contained in the ideal $\gcd\{S\}\mathbb{Z}[\tau]$. In such a case, if $\gcd\{S\} \neq 1$, the τ'-inflation does not generate all points of $\mathbb{Z}[\tau]$ contained in some acceptance window, therefore $S^{\vdash}$ is not a cut and project set. The identification with a one-dimensional quasicrystal requires an affine mapping ϕ, such that $\gcd\{\phi(S)\} = 1$.

Lemma 2.3. For $u, v \in \mathbb{Z}[\tau]$ and any $x \in \{u, v\}^{\vdash}$, we have $\gcd\{x, u\} = \gcd\{u, v\}$.

Proof. Clearly, it suffices to show that for any $u, v \in \mathbb{Z}[\tau]$, $\gcd\{u, v\} = \gcd\{u \vdash v, u\}$. Since the factorization of a number $x \in \mathbb{Z}[\tau]$ into primes in $\mathbb{Z}[\tau]$ is unique up to some divisor of 1, the greatest common divisor does not change, if we multiply the argument by any divisor of unity τ^k, with $k \in \mathbb{Z}$, $\gcd\{u, v\} = \gcd\{\tau^k u, v\}$. Clearly, $\gcd\{x, y\} = \gcd\{x - y, y\}$.

Therefore $\gcd\{u \vdash v, u\} = \gcd\{\tau^2 u - \tau v, u\} = \gcd\{\tau^2 u - \tau v, \tau^2 u\} = \gcd\{\tau v, \tau^2 u\} = \gcd\{v, u\}$. The statement of the lemma follows easily. $\square$

The aim of this paper is to identify any τ-inflation invariant Delone set Λ in $\mathbb{R}^n$, with a cut and project quasicrystal. The problem has several aspects.

First, the stage for a cut and project quasicrystal is a $\mathbb{Z}[\tau]$-lattice M, which is equipped with a star map $*$. Therefore our first task is to embed Λ into an M. However, as we have seen from lemma 2.3 for the one-dimensional case, not all affine mappings ϕ, $\phi(\Lambda) \subset M$, are suitable. In general, we require that $[\phi(\Lambda)]^\tau = M$.

Having found an embedding ϕ of Λ into a suitable $\mathbb{Z}[\tau]$-lattice, one can check the star map image of $\phi(\Lambda)$, in correspondence with the cut and project definition. The desired acceptance window is found as convex hull $\langle \phi^*(\Lambda) \rangle$.

The steps recalled above are rather natural, however, they need to be justified. First of all, is it possible to find a suitable embedding of Λ into a $\mathbb{Z}[\tau]$-lattice? The answer is yes, for Λ, which is τ-inflation invariant and Delone. When considering the acceptance window, do all the points of the $\mathbb{Z}[\tau]$-lattice contained in $\langle \phi^*(\Lambda) \rangle$, correspond to points of Λ, or are there some additional ones? It turns out that this is true for the interior of the window only. The question of the boundary should be dealt with in more detail.

The main result of this paper is formulated as the following theorem.

Theorem 2.4. Let $\Lambda \subset \mathbb{R}^n$, $n \geqslant 1$, be a Delone set closed under τ-inflation. There exists an affine mapping $\phi: \mathbb{R}^n \to \mathbb{R}^n$, a $\mathbb{Z}[\tau]$-lattice $M = \sum_{i=1}^n \mathbb{Z}[\tau]\alpha_i$ equipped with a star map, and a bounded set $\Omega \subset \mathbb{R}$, with the convex interior Ω°, such that

$$\phi(\Lambda) = \Sigma(\Omega).$$

The convexity of the interior Ω° of the acceptance window describe Ω partially. A necessary and sufficient condition on Ω to have $\Sigma(\Omega)$ τ-inflation invariant, is formulated using the notion of quasiconvexity (cf definition 5.4).

The subsequent parts of this paper contain the proof, comments and examples. Some of the auxiliary statements are of interest on their own. Particularly instructive is the one-dimensional case.

3. One-dimensional sets

In this section we demonstrate in full the one-dimensional special case of the main theorem as theorem 3.1. Most of the auxiliary steps are indispensable during the proof of the multidimensional case. Some properties of higher-dimensional quasicrystals, such as inflation symmetries [5], minimal distances [6], etc are in fact one-dimensional problems.

The quasiconvexity of Ω, in theorem 3.1, is replaced by the simple assertion that Ω is convex, due to the fact that its boundary is of dimension 0.

Theorem 3.1. Let $\Lambda \subset \mathbb{R}$ be a Delone set closed under τ-inflation. There exists an affine mapping $\phi : \mathbb{R} \to \mathbb{R}$ and a nondegenerated bounded interval $\Omega \subset \mathbb{R}$ such that

$$\phi(\Lambda) = \Sigma(\Omega).$$

The first step is to map $\Lambda \subset \mathbb{R}$ into $\mathbb{Z}[\tau]$. In fact not all mappings into $\mathbb{Z}[\tau]$ are suitable for subsequent identification of Λ with the quasicrystal.

For a set $\Lambda \subset \mathbb{Z}[\tau]$, containing 0 and closed under τ-inflation, lemma 2.3 implies that Λ is contained in $\gcd\{\Lambda\}\mathbb{Z}[\tau]$, which is an ideal in the ring $\mathbb{Z}[\tau]$. Consequently, for any such set Λ there exists an affine mapping ψ (multiplication by $(\gcd\{\Lambda\}^{-1})$, which makes the inclusion $\psi(\Lambda) \subset \xi\mathbb{Z}[\tau]$ to imply that ξ is a divisor of unity, i.e. $\xi\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$. After such an embedding into $\mathbb{Z}[\tau]$, Λ is a cut and project quasicrystal with a convex acceptance window, as shown in the following lemmas.

Lemma 3.2. Let $\Lambda \subset \mathbb{R}$ be a Delone set closed under τ-inflation. Then there exist an affine mapping $\phi: \mathbb{R} \to \mathbb{R}$ such that $\phi(\Lambda) \subset \mathbb{Z}[\tau]$ and $0, 1 \in \phi(\Lambda)$.

Proof. Take any $b \in \Lambda$. Since Λ is Delone, the greatest $a \in \Lambda$ smaller than b, and the smallest $c \in \Lambda$ greater than b, are well determined. Thus we consider points $a < b < c$ adjacent in Λ. In the first step we show that for any $b \in \Lambda$ the ratio of length of the line segments bounded by a, b and c can only take certain values. Namely,

$$\frac{b-a}{c-b} = \tau, 1, \text{ or } \frac{1}{\tau}. \tag{8}$$

Since the τ-inflation invariance is preserved under an affine mapping, without loss of generality we assume that $a = -1$, $b = 0$ and $c \geqslant 1$. If $c = 1$ or $c = \tau$, then (8) is fulfilled. Suppose that $\tau \neq c > 1$. Clearly, $0 < c < \tau$. Indeed, if $c > \tau$, then $0 \vdash (-1) = \tau < c$, which is the contradiction with the assumption that 0 and c are adjacent.

Since both c and τ belong to Λ, also $u \equiv c \vdash \tau = \tau^2 c - \tau^2 = \tau^2(c-1) \in \Lambda$. However, $0 < u < c$, which is again the contradiction with the fact that 0 and c are adjacent in Λ.

For arbitrary point b and its left neighbour $a < b$ we now put $\phi(a) = 0$, $\phi(b) = 1$. The affine mapping is now well determined. Since $\phi(a), \phi(b) \in \mathbb{Z}[\tau]$ and all distances between adjacent points of Λ in this scale are of the form τ^k, $k \in \mathbb{Z}$, we have $\phi(\Lambda) \subset \mathbb{Z}[\tau]$. □

Since $0, 1 \in \phi(\Lambda) \subset \mathbb{Z}[\tau]$, we have $\gcd\{\phi(\Lambda)\} = 1$. In order to identify $\phi(\Lambda)$ with a cut and project quasicrystal, we find its acceptance interval as the convex hull of the star map image $\phi^*(\Lambda) = \phi'(\Lambda)$ of $\phi(\Lambda)$. Then we have to show that the only points from $\mathbb{Z}[\tau]$ in the convex hull are the elements of $\phi'(\Lambda)$.

Let us introduce the τ-expansion, an important tool used several times later (lemma 3.3, proof of theorem 3.1). A *τ-expansion* [11] of a real number $x \geqslant 0$ is an infinite sequence $(x_i)_{k \geqslant i \geqslant \infty}$ of coefficients taking only two values 0 and 1, so that one has the equality

$$x = \sum_{i=-\infty}^{k} x_i \tau^i \qquad x_i = 0 \text{ or } 1. \tag{9}$$

The identity $\tau^j + \tau^{j+1} = \tau^{j+2}$, $j \in \mathbb{Z}$, implies that $x_i x_{i-1} = 0$, $k \geqslant i \in \mathbb{Z}$. For negative real numbers we put $x_i = -|x|_i$, negatives of τ-expansion coefficients for $|x|$.

If an expansion ends with infinitely many zeros, it is said to be *finite*, and the zeros at the end are omited. The set $\mathrm{Fin}(\tau)$ of all numbers with finite τ-expansion coincides with the ring $\mathbb{Z}[\tau]$. In particular, the τ-expansion of any $x \in \mathbb{Z}[\tau] \cap (0, 1)$ contains only negative powers of τ,

$$x = \sum_{i=1}^{k} \alpha_i \tau^{-i} = \sum_{i=1}^{k} \alpha_i \frac{1}{\tau^i} \qquad \alpha_i \in \{0, 1\}; \alpha_i \alpha_{i+1} = 0.$$

The following lemma is proven in [2]. Its demonstration below follows a different path, and it is a crucial result for our purposes.

Lemma 3.3.

$$\mathbb{Z}[\tau] \cap [0, 1] = \{0, 1\}^{-1}.$$

Proof. The inclusion '$\supset$' is obvious; a combination of 0 and 1 with the coefficients $(1/\tau)^2$ and $1/\tau$ is convex, therefore must lie between 0 and 1. As for the opposite inclusion '$\subset$', clearly both 0 and 1 are included in $\{0, 1\}^{\dashv}$. Indeed, $x \dashv x = x$. For $x \in (0, 1)$ we use its τ-expansion. We will proceed recursively on the maximal power of $\frac{1}{\tau}$ in the τ-expansion of the number x, denoted by k:

(1) $k = 1$:

$$x = \frac{1}{\tau} = \frac{0}{\tau^2} + \frac{1}{\tau} \in \{0, 1\}^{\dashv}$$

(2) $k = 2$:

$$x = \frac{1}{\tau^2} = \frac{1}{\tau^2} + \frac{0}{\tau} \in \{0, 1\}^{\dashv}$$

(3) $k \geqslant 3$:

$$x = \frac{\alpha_1}{\tau} + \frac{\alpha_2}{\tau^2} + \cdots + \frac{\alpha_k}{\tau^k}$$

(a) either:

$$\alpha_1 = 0 \Longrightarrow x = \frac{0}{\tau^2} + \frac{1}{\tau}\left(\frac{\alpha_2}{\tau} + \cdots + \frac{\alpha_k}{\tau^{k-1}}\right) \in \{0, 1\}^{\dashv}$$

(b) or:

$$\alpha_1 = 1 \Rightarrow \alpha_2 = 0 \Longrightarrow x = \frac{1}{\tau^2}\left(\frac{\alpha_3}{\tau} + \cdots + \frac{\alpha_k}{\tau^{k-2}}\right) + \frac{1}{\tau} \in \{0, 1\}^{\dashv}.$$

□

The following remark, as a consequence of the lemma above, will be used in section 5.

Remark 3.4. A set $\Lambda \subset \mathbb{R}$, containing at least two elements and closed under τ-inflation, which is not Delone, is dense in $\mathbb{R}$.

The content of lemma 3.3 is readily extended to any interval $[u, v]$ with $u, v \in \mathbb{Z}[\tau]$, such that $v - u = \tau^k$ for some integer k. Using only properties of quasiaddition and the fact that τ^k is a divisor of unity in the ring $\mathbb{Z}[\tau]$, i.e. $\tau^{-k}\mathbb{Z}[\tau] = \mathbb{Z}[\tau]$, we obtain the following result.

Corollary 3.5. Let $u, v \in \mathbb{Z}[\tau]$, such that $v - u = \tau^k$, for some $k \in \mathbb{Z}$. Then

$$\{u, v\}^{\dashv} = [u, v] \cap \mathbb{Z}[\tau].$$

Unlike the previous case, if the difference $v - u$ is not a unit in $\mathbb{Z}[\tau]$, not all the points of $\mathbb{Z}[\tau]$ in the interval $[u, v]$ can be generated by τ'-inflation. However, using the operation $\dashv$ we obtain all points in the interval of a certain ideal in $\mathbb{Z}[\tau]$. Let us see an example. Let $\Lambda = \{0, 3\}^{\dashv}$. Suppose there exists a convex set (interval) $P \subset \mathbb{R}$, such that $\Lambda = P \cap \mathbb{Z}[\tau]$. Since $\Lambda = \{0, 3\}^{\dashv}$, we have $0, 3 \in \Lambda$. Therefore the interval P contains the interval $[0, 3]$. However, this means that for example $1 \in \mathbb{Z}[\tau] \cap [0, 3]$ should belong to Λ. Clearly, due to lemma 2.3, we have $1 = \gcd\{1, 3\} \neq \gcd\{0, 3\} = 3$. Therefore $1 \notin \Lambda$. In general, the description of τ'-inflation closure of a finite subset of $\mathbb{Z}[\tau]$ can be written as follows.

Lemma 3.6. Let $a_0 < a_1 < a_2 < \cdots < a_n$, $a_i \in \mathbb{Z}[\tau]$ for $i = 0, \ldots, n$. Then

$$\{a_0, a_1, \ldots, a_n\}^{\dashv} = [a_0, a_n] \cap (\gcd\{a_1 - a_0, \ldots, a_n - a_0\}\mathbb{Z}[\tau] + a_0).$$

Note that a set $\Lambda \subset \mathbb{R}$, containing at least two elements and closed under τ'-inflation, is dense in $\langle\Lambda\rangle$, the convex hull of Λ.

Occasionally a quasicrystal can be constructed as a τ-inflation closure of finite set of seed points. Far from every set of seed points leads to a Delone set. The following assertion can guide the choice of seeds.

Corollary 3.7. Let S be a finite subset of $\mathbb{R}$ containing 0 and at least two additional points. Then $S^{\vdash}$ is Delone if and only if there exists an $x \in S$, such that all elements of S are $\mathbb{Q}[\tau]$-multiples of x. In particular, let $a, b, c \in \mathbb{R}$. The set $\{a, b, c\}^{\vdash}$ is Delone if and only if

$$\frac{c-b}{b-a} \in \mathbb{Q}[\tau]. \tag{10}$$

Otherwise $\{a, b, c\}^{\vdash}$ is dense in $\mathbb{R}$.

Proof. We prove the statement for S with three elements, the general assertion then follows easily. Let $a, b, c \in \mathbb{R}$. The implication ($\Rightarrow$) can be seen from lemma 3.2. For the exists an $x \in S$, such that all elements of S are $\mathbb{Q}[\tau]$-multiples other implication, suppose we have points $a < b < c \in \mathbb{R}$ satisfying (10). Since the action of a linear mapping does not change the Delone property, we can consider a scale such that $a = 0$, $b = 1$. Then necessarily $1 < c \in \mathbb{Q}[\tau]$, i.e. there exists $p, q \in \mathbb{Z}[\tau]$, $\gcd\{p, q\} = 1$ such that $c = \frac{p}{q}$. By another rescaling we encounter the problem whether the set $\{0, q, p\}^{\vdash}$ is Delone or not. We use lemma 3.6 for the set of points $S = \{0, q', p'\}$. Since $\gcd\{q', p'\} = 1$, we have $\{0, q', p'\}^{\dashv} = [\min\{0, q', p'\}, \max\{0, q', p'\}] \cap \mathbb{Z}[\tau]$, which is equivalent to $\{0, q, p\}^{\vdash} = \Sigma\left([\min\{0, q', p'\}, \max\{0, q', p'\}]\right)$, i.e. it is a cut and project quasicrystal with bounded acceptance domain. It is shown in [9] that such a set is Delone. $\square$

A multidimensional analogue of (10) is in corollary 5.2.

The following lemma shows that the Delone property of $\Sigma(\Omega)$ is lost when Ω is not bounded.

Lemma 3.8. Let $\Sigma(\Omega) = \{x \in \mathbb{Z}[\tau] \mid x' \in \Omega\}$ be a Delone set for an interval $\Omega \subset \mathbb{R}$. Then Ω is non degenerated and bounded.

Proof. If Ω is degenerated, i.e. $\Sigma(\Omega)$ is either empty or contains only one element, then it is not relatively dense and hence not Delone (cf definition 2.1). An unbounded connected Ω implies that there exists an $\alpha \in \mathbb{R}$ such that either $(\alpha, +\infty)$, or $(-\infty, \alpha)$ is subset of Ω. Let $(\alpha, +\infty) \subset \Omega$, the other case would be treated analogically. Without loss of generality, $\alpha > 0$. We want to obtain a contradiction with uniform discreteness of $\Sigma(\Omega)$. To prove its negation,

$$(\forall\varepsilon > 0)(\exists x, y \in \Sigma(\Omega))(|y - x| < \varepsilon) \tag{11}$$

we take $k \in \mathbb{Z}$ such that $\tau^{k+2} > \tau^k > \max\{\alpha, \frac{1}{\varepsilon\tau}\}$. Since $\tau^{k+2} > \tau^k > \alpha$, we have $\tau^{k+2}, \tau^k \in \Omega$, hence $(\tau^{k+2})', (\tau^k)' \in \Sigma(\Omega)$. Putting $x = (\tau^k)'$ and $y = (\tau^{k+2})'$, we find (11) as

$$|(\tau^{k+2})' - (\tau^k)'| = \frac{1}{\tau^k} - \frac{1}{\tau^{k+2}} = \frac{1}{\tau^{k+1}} < \frac{1}{\tau}\tau\varepsilon = \varepsilon.$$

Therefore $\Sigma(\Omega)$ is not uniformly discrete, hence not Delone. $\square$

Now we are in the position to prove the one-dimensional version (theorem 3.1) of the main theorem.

Proof of theorem 3.1. Using lemma 3.2, we have a linear mapping $\phi : \mathbb{R} \to \mathbb{R}$ such that $0, 1 \in \phi(\Lambda) \subset \mathbb{Z}[\tau]$. This means that $\gcd\{\phi(\Lambda)\} = 1$. Denote by Ω the convex hull of the star map image of $\phi(\Lambda)$, $\Omega := \langle \phi^*(\Lambda) \rangle$. First we prove that $\Omega \cap \mathbb{Z}[\tau] = \phi^*(\Lambda)$.

The inclusion $\supset$ is obvious. Let us look at the opposite one: $\subset$. For any $x \in \Omega \cap \mathbb{Z}[\tau]$ we want to show that $x \in \phi^*(\Lambda)$. Using corollary 3.5, we conclude that it suffices to find an interval $[c, d] \subset \Omega$ of unit length, bounded by points from $\phi^*(\Lambda)$. Clearly, $[0, 1] \subset \Omega$. Using lemma 3.3, we have $([0, 1] \cap \mathbb{Z}[\tau]) \subset \phi^*(\Lambda)$.

Suppose that Ω is bounded by some $-\infty \leqslant u, v \leqslant +\infty$. The case $u = 0$, $v = 1$ has already been established. Suppose that $v > 1$. Let $x \in \Omega \cap \mathbb{Z}[\tau]$, $x > 1$. Since $\phi^*(\Lambda)$ is dense in Ω, there exists an element $w \in \phi^*(\Lambda)$, such that $x < w < v$. Since w is an element of $\mathbb{Z}[\tau]$, the number $w - 1 \neq 0$ has the finite τ-expansion,

$$w - 1 = \tau^l + \sum_{i=-j}^{l-2} w_i \tau^i.$$

Clearly, $w - \tau^l < 1 + \tau^l$. Since $\phi^*(\Lambda)$ is dense in Ω, we can choose a $d \in \phi^*(\Lambda)$ such that $\max\{w - \tau^l, \tau^l\} < d < 1 + \tau^l$. Then necessarily $c := d - \tau^l$ belongs to $[0, 1] \cap \mathbb{Z}[\tau]$, hence also to $\phi^*(\Lambda)$. The length of the interval $[c, d]$ is τ^l, therefore $\phi^*(\Lambda) \supset \{c, d\}^{\dashv} = [c, d] \cap \mathbb{Z}[\tau]$. Now it suffices to take the interval $[w - \tau^l, w]$, where $w - \tau^l \in [c, d]$, to obtain $x \in ([0, w] \cap \mathbb{Z}[\tau]) \subset \phi^*(\Lambda)$. Similarly we proceed for an element $x \in \Omega \cap \mathbb{Z}[\tau]$, $u < x < 0$. Finally, $\phi^*(\Lambda) \supset \Omega \cap \mathbb{Z}[\tau]$. Thus we have

$$\phi(\Lambda) = \Sigma(\Omega).$$

To prove the statement of theorem 3.1, it suffices to notice that, due to lemma 3.8, Ω is bounded. $\square$

4. Generators of inflation invariant sets

In this section we have collected several lemmas which subsequently facilitate the proof of the main theorem 2.4. We describe the τ'-inflation closure $S^{\dashv}$ of a set S of generators ('seed points'). Lemma 4.1 and corollary 4.2 describe the case of the set generated from two seed points, $\{x, y\}^{\dashv}$. In lemma 4.4 the set of generators is taken to be a basis of a $\mathbb{Z}[\tau]$-module. The result for an arbitrary set of generators is given in lemma 4.6.

Lemma 4.1. Let $x, y \in M \subset \mathbb{R}^n$ be two points in a $\mathbb{Z}[\tau]$-module. Then

$$\{x, y\}^{\dashv} = \{x + s(y - x) \mid s \in \mathbb{Z}[\tau] \cap [0, 1]\}.$$

Corollary 4.2 singles out the case when τ'-inflation of two elements $x, y \in M$ generates all points of the $\mathbb{Z}[\tau]$-module on the line segment $\langle x, y \rangle$.

Corollary 4.2. Let x, y be two points in a $\mathbb{Z}[\tau]$-module $M \subset \mathbb{R}^n$, with the basis α_i, $x = \sum x_i \alpha_i$ and $y = \sum y_i \alpha_i$, which, at least for one $i \in \{1, \ldots, n\}$, satisfy $y_i - x_i = \tau^k$. Then

$$\{x, y\}^{\dashv} = \{x + s(y - x) \mid s \in [0, 1]\} \cap M.$$

Our ultimate aim is to describe the cut and project scheme of a Delone, τ-inflation invariant set $\Lambda \subset \mathbb{R}^n$. The first task is to embed Λ into a suitable $\mathbb{Z}[\tau]$-module M. Only then can we ask about its acceptance window. A hint for finding the window is a direct consequence of lemma 4.1.

Let $\Lambda \subset M \subset \mathbb{R}^n$ be a Delone set, closed under τ-inflation. Then there exists a convex set $\Omega \subset \mathbb{R}^n$,

$$\Omega := \langle \Lambda^* \rangle \tag{12}$$

such that Λ^* is dense in Ω. Here $\langle \Lambda^* \rangle$ denotes the convex hull of Λ^*. Moreover, Λ^* is dense on any line segment bounded by points of Λ^*.

In the case that $\langle \Lambda^* \rangle \cap M^* = \Lambda^*$, the set Λ is a quasicrystal $\Sigma(\Omega)$, with the convex acceptance window Ω given by (12). The following lemma helps us to decide the validity of $\langle \Lambda^* \rangle \cap M^* = \Lambda^*$.

Lemma 4.3. Let $\Lambda \subset M$ be a Delone set, closed under τ-inflation, and Ω given by (12). If there exists a non-empty open set $\widetilde{\Omega} \subset \Omega$, such that $\widetilde{\Omega} \cap M^* \subseteq \Lambda^*$. Then

$$\Omega^\circ \cap M^* \subseteq \Lambda^*$$

where Ω° is the interior of the set Ω.

Proof. We have to show that any $z \in \Omega^\circ \cap M^*$ is contained in Λ^*.

First observe that for any point $x \in \Lambda^*$ and any $y \in \widetilde{\Omega} \cap M^*$, all points z of the $\mathbb{Z}[\tau]$-module M^*, on the line segment bounded by x and y, belong to Λ^*. Using corollary 4.2, we can view the situation in one of the coordinates and thus face a one-dimensional problem solved in the proof of theorem 3.1.

Now, let $z \in \Omega^\circ \cap M^*$. Choose arbitrary $b \in \widetilde{\Omega} \cap M^*$ and $c \in \Omega^\circ$, such that z lies on the line segment between b and c (see figure 2(a)). Since Λ^* is dense in Ω, it is possible to find a point $x \in \Lambda^*$ sufficiently closed to c, such that the straight line through z, x intersects $\widetilde{\Omega}$ nontrivially. Since $\widetilde{\Omega}$ is open, one can choose a point $y \in M^*$ in this intersection. Then z lies on the line segment between y and x (see figure 2(b)). Using the observation, we find $z \in \Lambda^*$. □

In lemma 4.4 we show that all $\mathbb{Z}[\tau]$-module points inside a certain polytop are generated by τ'-inflation starting from its vertices.

Lemma 4.4. Let $M \subset \mathbb{R}^n$ be a $\mathbb{Z}[\tau]$-module with a basis α_i, $i = 1, \ldots, n$. Then

$$\{0, \alpha_1, \ldots, \alpha_n\}^{\dashv} = M \cap \langle 0, \alpha_1, \ldots, \alpha_n \rangle.$$

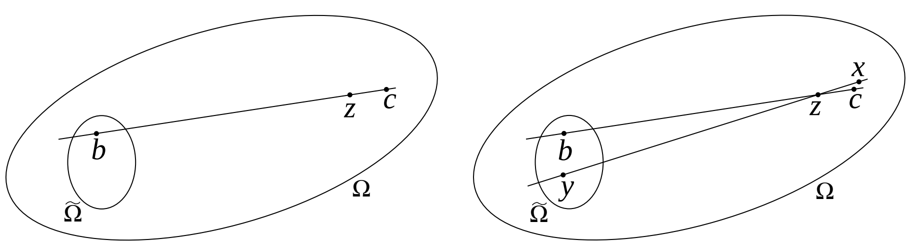

Figure 2. A symbolic representation of two subsequent steps of the proof of lemma 4.3.

Proof. We show that all points of the module M of the polytope with vertices $0, \alpha_1, \ldots, \alpha_n$ (convex combinations of the vertices), can be generated by τ'-inflation, using the vertices as generators. We proceed recursively on the number of nonzero coefficients in the convex combination.

Using lemma 4.1 and corollary 4.2, we obtain

$$\{0, \alpha_i\}^{\dashv} = (\mathbb{Z}[\tau] \cap [0, 1))\alpha_i = M \cap \langle 0, \alpha_i \rangle \tag{13}$$

where we denote using $\langle 0, \alpha_i \rangle$ the convex hull of vertices 0, α_i, i .e. the line segment with boundary points 0, α_i. Similarly,

$$\{\alpha_i, \alpha_j\}^{\dashv} = (\mathbb{Z}[\tau] \cap [0, 1))(\alpha_j - \alpha_i) + \alpha_i = M \cap \langle \alpha_i, \alpha_j \rangle. \tag{14}$$

Now, let the point $z \in M$ be the convex combination of $k + 1$ vertices of the polytope, for some $2 < k + 1 \leqslant n + 1$. Without loss of generality suppose that

$$z = z_0.0 + \sum_{i=1}^{k} z_i \alpha_i \qquad z_i \in \mathbb{Z}[\tau], z_i > 0 \qquad i = 0, \ldots, k \qquad \sum_{i=1}^{k} z_i < 1.$$

First, let us consider a point z with the property

$$z_1 < \frac{1}{\tau} \qquad \text{and} \qquad \sum_{i=2}^{k} z_i < \frac{1}{\tau^2}. \tag{15}$$

Then

$$z = z_1\alpha_1 + \sum_{i=2}^{k} z_i \alpha_i = \frac{1}{\tau}(\tau z_1 \alpha_1) + \frac{1}{\tau^2}\left(\sum_{i=2}^{k} \tau^2 z_i \alpha_i \right)$$

which is, due to induction hypothesis, a τ'-inflation combination of two points $\tau z_1 \alpha_1 \in \{0, \alpha_1\}^{\dashv}$ and $\sum_{i=2}^{k} \tau^2 z_i \alpha_i \in \{0, \alpha_1, \ldots, \alpha_k\}^{\dashv}$ of the polytope. Thus $z \in \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}^{\dashv}$.

It now suffices to realize that the region of points z, satisfying (15), defines an open (with respect to $\mathbb{R}^k$) set $\widetilde{\Omega} \subset \langle 0, \alpha_1, \ldots, \alpha_n \rangle =: \Omega$. Due to lemma 4.3, applied to the $\mathbb{Z}[\tau]$-lattice $M^{(k)} := \sum_{i=1}^{k} \mathbb{Z}[\tau]\alpha_i$, we have

$$\langle 0, \alpha_1, \ldots, \alpha_k \rangle^{\circ} \cap M = \langle 0, \alpha_1, \ldots, \alpha_k \rangle^{\circ} \cap M^{(k)} \subset \{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}^{\dashv},$$

where all interiors are meant with respect to $\mathbb{R}^k$.

Boundary points of $\langle 0, \alpha_1, \ldots, \alpha_k \rangle^{\circ} \cap M$ are in $\{0, \alpha_1, \alpha_2, \ldots, \alpha_k\}^{\dashv}$, according to induction hypothesis. $\square$

For the following corollary realize that if M is a $\mathbb{Z}[\tau]$-lattice with basis α_i, then corresponding star map image M^* is also a $\mathbb{Z}[\tau]$-lattice. Its basis is α_i^*.

Corollary 4.5. Let $M \subset \mathbb{R}^n$ be a $\mathbb{Z}[\tau]$-module with a basis α_i, $i = 1, \ldots, n$, equipped with a star map. The set $\{0, \alpha_1, \ldots, \alpha_n\}^{\vdash}$ is a cut and project quasicrystal with acceptance window $\langle 0, \alpha_1^*, \ldots, \alpha_n^* \rangle$.

The following lemma is used in the proof of lemma 5.1 and implies the proposition 5.4.

Lemma 4.6. Let $S \subset M$. Then

$$[S]^{\tau} \cap \langle S \rangle^{\circ} \subset S^{\dashv}.$$

Proof. Without loss of generality assume that $0 \in S$. In this case

$$[S]^\tau = \left\{ \sum_{i=1}^{m} \xi_i u_i \;\middle|\; \xi_i \in \mathbb{Z}[\tau], u_i \in S \right\}$$

coincides with the $\mathbb{Z}[\tau]$-submodule of M generated by S.

Suppose initially that S is finite. We proceed recursively on the number of elements in S. Let $u \in M$, $[0, u]^\tau := \{tu \mid t \in \mathbb{Z}[\tau]\}$. Then using corollary 4.2,

$$[0, u]^\tau \cap \langle 0, u\rangle := \{tu \mid t \in \mathbb{Z}[\tau] \cap [0, 1]\} = \{0, u\}^{\dashv}.$$

Suppose now, that for any $S \subset M$, with less than $k + 1$ elements, the statement holds. Take $S = \{0, u_1, \ldots, u_k\}$, where $u_i \in M$. We show that

$$[0, u_1, \ldots, u_k]^\tau \cap \langle 0, u_1, \ldots, u_k\rangle^\circ \subset \{0, u_1, \ldots, u_k\}^{\dashv}.$$

Consider $u \in [0, u_1, \ldots, u_k]^\tau \cap \langle 0, u_1, \ldots, u_k\rangle^\circ$. We can write

$$u = \sum_{i=1}^{k} \beta_i u_i \qquad \beta_i \in \mathbb{R}, \beta_i > 0 \qquad \sum_{i_1}^{k} \beta_i < 1.$$

Assume that

$$\beta_i \in \mathbb{Z}[\tau] \qquad \forall i \in \{1, \ldots, k\}. \tag{16}$$

For linearly dependent u_i's, the assumption is not obvious. It is justified in the final part of this proof. Given assumption (16) and using the same procedure as in the proof of lemma 4.4, we consider first the points $u = \sum \beta_i u_i$ satisfying

$$\beta_1 < \frac{1}{\tau} \qquad \text{and} \qquad \sum_{i=2}^{n} \beta_i < \frac{1}{\tau^2}.$$

Then u is a τ'-inflation combination of points which are, due to induction hypothesis, already in $\{0, u_1, \ldots, u_k\}^{\dashv}$. These points form an open set $\widetilde{\Omega}$ for lemma 4.3, which gives us the result.

It now suffices to show the statement also for infinite sets S. Let $x \in [S]^\tau \cap \langle S\rangle^\circ$. There exists a finite set $P \subset S$, such that $x \in [P]^\tau \cap \langle P\rangle^\circ$, which we have shown to be a subset of $\{P\}^{\dashv} \subset \{S\}^{\dashv}$.

Let us now justify assumption (16). Since $u \in [0, u_1, \ldots, u_k]^\tau \cap \langle 0, u_1, \ldots, u_k\rangle^\circ$, we can find $\delta_1, \ldots, \delta_k \in \mathbb{Z}[\tau]$, such that

$$u = \sum_{i=1}^{k} \delta_i u_i \tag{17}$$

and $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$, such that

$$u = \sum_{i=1}^{k} \gamma_i u_i \qquad \text{with } \gamma_i > 0 \qquad \sum_{i=1}^{k} \gamma_i < 1. \tag{18}$$

In the case of linearly independent $u_1, \ldots, u_k$, the coefficients δ_i and γ_i must coincide and we can put $\beta_i := \delta_i = \gamma_i$.

Otherwise there are many k-tuples $(\delta_1, \ldots, \delta_k)$ and $(\gamma_1, \ldots, \gamma_k)$ satisfying (17) and (18), respectively. Without loss of generality, we assume that $u_1, \ldots, u_l$ are linearly independent for $l < k$, and l is the rank of $\{u_1, \ldots, u_k\}$. Since $u_{l+1}, \ldots, u_k$ are dependent on $u_1, \ldots, u_l$, there exists coefficients α_r and α_i^r, for $r = l + 1, \ldots, k$, and $i = 1, \ldots, l$, such that

$$\alpha_r u_r = \sum_{i=1}^{l} \alpha_i^r u_i \qquad r = l + 1, \ldots, k. \tag{19}$$

Clearly, the coefficients α_r and α_i^r can be chosen from $\mathbb{Z}[\tau]$.

We can add a zero to equation (17), given by sum of relations (19),

$$0 = \sum_{r=l+1}^{k} \left[\sum_{i=1}^{l} \alpha_i^r u_i - \alpha_r u_r \right]$$

multiplied by any factor $H_r \in \mathbb{Z}[\tau]$,

$$u = \sum_{i=1}^{l} \left(\delta_i + \sum_{r=l+1}^{k} H_r \alpha_i^r \right) u_i + \sum_{i=l+1}^{k} (\delta_i - H_i \alpha_i) u_i. \tag{20}$$

Subtracting (18) from (20), one has

$$u - u = 0 = \sum_{i=1}^{l} \left(\delta_i - \gamma_i + \sum_{r=l+1}^{k} H_r \alpha_i^r \right) u_i + \sum_{i=l+1}^{k} (\delta_i - \gamma_i - H_i \alpha_i) u_i. \tag{21}$$

Using the free parameter $H_i \in \mathbb{Z}[\tau]$, $i = l+1, \ldots, k$, and the fact that $\mathbb{Z}[\tau]$ is dense in $\mathbb{R}$, it is possible, for any $\varepsilon > 0$, to find $H_i \in \mathbb{Z}[\tau]$ such that the coefficients of u_i, $i = l+1, \ldots, k$, in (21) are close to zero,

$$|\delta_i - H_i \alpha_i - \gamma_i| < \varepsilon.$$

From (21), one obtains

$$z := \sum_{i=l+1}^{k} (\gamma_i - \delta_i + H_i \alpha_i) u_i = \sum_{i=1}^{l} \left(\delta_i - \gamma_i + \sum_{r=l+1}^{k} H_r \alpha_i^r \right) u_i$$

where $\|z\| < \varepsilon k \max\{\|u_i\| \mid i = 1, \ldots, k\}$. Since the coordinate functional is continuous, for given $\omega > 0$, one finds $\varepsilon > 0$ small enough, such that

$$\left| \delta_i + \sum_{r=l+1}^{k} H_r \alpha_i^r - \gamma_i \right| < \omega \qquad i = 1, \ldots, l \tag{22}$$

$$|\delta_i - H_i \alpha_i - \gamma_i| < \varepsilon \qquad i = l+1, \ldots, k. \tag{23}$$

Now put

$$\beta_i := \begin{cases} \delta_i + \sum_{r=l+1}^{k} H_r \alpha_i^r & \in \mathbb{Z}[\tau] \qquad i = 1, \ldots, l \\ \delta_i - H_i \alpha_i & \in \mathbb{Z}[\tau] \qquad i = l+1, \ldots, k. \end{cases}$$

One can choose ω and ε in (22) and (23) small enough, such that β_i have the same properties as γ_i in (18), i.e. satisfy (16). □

A consequence of the above lemma is formulated as corollary 4.7. It is a very important step for the proof of the main theorem, namely for identification of the acceptance window of the inflation closure of an arbitrary set $S \subset M$ of generators.

Corollary 4.7. Let S be a subset of a $\mathbb{Z}[\tau]$-lattice M, equipped with a star map, such that $[S]^\tau = M$. Then

$$\Sigma(\langle S^* \rangle^\circ) \subset S^\vdash \subset \Sigma(\langle S^* \rangle).$$

The above corollary identifies the inflation closure of an arbitrary set of generators with a cut and project set. If the assumption $[S]^\tau = M$ is not valid, we consider as the corresponding $\mathbb{Z}[\tau]$-lattice the set $[S]^\tau - a_0$, for some $a_0 \in S$, where the star map is induced from M.

There are two important facts to notice. First, corollary 4.7 does not deal with the Delone property of the inflation closure $S^{\vdash}$. Therefore no statement about boundedness of the acceptance window can be made. Therefore in general, the resulting set may not be a quasicrystal. Secondly, we determine only the interior of the acceptance window. In order to provide complete information about its boundary, we introduce the notion of quasiconvexity (definition 5.4). The following section gives the answers for both of the problems.

5. Proof of the main theorem

Herein the proof of theorem 2.4 is completed. However, before that several essential steps have to be taken, some of them being of wider interest. Let us point some of them out. Similarly as in the one-dimensional case, an arbitrary τ-inflation invariant Delone set in $\mathbb{R}^n$ needs to be mapped into a $\mathbb{Z}[\tau]$-lattice, in order to be identified with a quasicrystal (lemma 5.1). A particularly useful for generating quasicrystals is a consequence of the lemma, which is formulated as corollary 5.2 (compare with corollary 3.7). Lemma 5.3 states that the boundedness of Ω is not only a sufficient, but also a necessary condition for a cut and project quasicrystal closed under τ-inflation, to be Delone. The quasiconvexity defined at the end of the section, allows us to formulate the general necessary and sufficient condition for $\Sigma(\Omega)$ to be closed under quasiaddition (proposition 5.5).

The mapping of an arbitrary Delone τ-inflation invariant set into a $\mathbb{Z}[\tau]$-lattice is accomplished according to lemma 5.1.

Lemma 5.1. Let $\Lambda \subset \mathbb{R}^n$ be a Delone set closed under τ-inflation. There exists a basis $\alpha_i \in \mathbb{R}^n$, such that Λ can be embedded using an affine mapping $\phi: \mathbb{R}^n \to \mathbb{R}^n$ in the $\mathbb{Z}[\tau]$-lattice $M := \sum_{i=1}^n \mathbb{Z}[\tau]\alpha_i$, $\phi(\Lambda) \subset M$, and $0, \alpha_i \in \phi(\Lambda)$.

Note that the basis of M belongs to $\phi(\Lambda)$, which implies that $\phi(\Lambda) \subset M$ is not contained in any proper submodule of M.

Proof of lemma 5.1. Initially we find an affine embedding $\phi: \mathbb{R}^n \to \mathbb{R}^n$ of Λ into a $\mathbb{Z}[\tau]$-module L. Let β_i for $i = 1, \ldots, n$ be a basis of $\mathbb{R}^n$, such that $(\beta_i \mid \beta_j) \in \mathbb{Z}[\tau]$ for all i, j. Denote by L the $\mathbb{Z}[\tau]$-module generated by the basis β_i, $L := \sum_{i=1}^n \mathbb{Z}[\tau]\beta_i$. Without loss of generality suppose that $0 \in \Lambda$. Since Λ is relatively dense, it spans $\mathbb{R}^n$ and hence there exists $n+1$ elements $0 \equiv x_0, x_1, \ldots, x_n \in \Lambda$, such that $x_1, \ldots, x_n$ are linearly independent. Let us distinguish the following possibilities.

(1) All coordinates of any point $y \in \Lambda$, relative to the basis x_i, belong to $\mathbb{Z}[\tau]$. Then put $\phi(x_i) = \beta_i$ for $i = 1, \ldots, n$.

(2) For any $j = 1, \ldots, n$, the set of jth coordinates of all points of Λ, relative to the basis x_i, are elements of $\frac{1}{p_j}\mathbb{Z}[\tau]$, for some $p_j \in \mathbb{Z}[\tau]$. Then put $\phi(x_i) = p_i\beta_i$ for $i = 1, \ldots, n$.

(3) If (1) and (2) are not fulfilled, say for first coordinate, we show the contradiction with the assumption that Λ is Delone. First, we find a sequence of points

$$\sum_{i=1}^n u_i^{(m)} x_i = u^{(m)} \in \Lambda \qquad \text{such that } u_1^{(m)} \overset{m\to\infty}{\longrightarrow} 0. \tag{24}$$

Without loss of generality, $u_1^{(m)}$ are mutually distinct. There are two possibilities.

(a) There exists a point $y \in \Lambda$ such that one of its coordinates, say the first one, in the basis $x_1, \ldots, x_n$ is not an element of $\mathbb{Q}[\tau]$. Now due to corollary 3.7, the set of first coordinates of $\{x_0, x_1, y\}^{\vdash}$ is dense everywhere in $\mathbb{R}$, thus 0 is its accumulation point. Since Λ is closed under quasiaddition, there exists a sequence of points satisfying (24).

(b) There exists an infinite sequence of $p^{(m)} \in \mathbb{Z}[\tau]$ and a sequence of points $y^{(m)} \in \Lambda$ such that for the first coordinate in the basis $x_1, \ldots, x_n$, there is $y_1^{(m)} \in \frac{1}{p^{(m)}}\mathbb{Z}[\tau]$ and $y_1^{(m)} \notin \frac{1}{p^{(k)}}\mathbb{Z}[\tau]$ for $k < m$. The τ-inflation closure $\{y_1^{(m)} \mid m \in \mathbb{N}\}^{\vdash}$ is a set closed under quasiaddition which cannot be embedded into $\mathbb{Z}[\tau]$, therefore according to lemma 3.2, is not Delone. Hence, due to remark 3.4, the point 0 is its accumulation point and we can construct the sequence $u^{(m)}$ of (24).

Put now $\phi(x_i) = \beta_i$, for $i = 2, \ldots, n$ and $\beta_i^* = \beta_i$. Let us denote

$$z^{(m)} := \sum_{i=2}^{n} u_i^{(m)} \beta_i.$$

Since $\{0, \beta_2, \ldots, \beta_n\}^{\vdash}$ is a cut and project quasicrystal (remark 4.5) in $\mathbb{R}^{n-1}$ with bounded acceptance window, it is Delone in $\mathbb{R}^{n-1}$ [9], i.e. relatively dense. There exists $r > 0$, such that for any $m \in \mathbb{N}$ there exists an element $w^{(m)} \in \{0, \beta_2, \ldots, \beta_n\}^{\vdash}$ such that

$$\|\tau z^{(m)} - w^{(m)}\| < r.$$

Take the sequence of points $u^{(m)} \vdash w^{(m)}$. The points are mutually distinct, since their first coordinates are $\tau^2 u_1^{(m)}$. One has

$$\|u^{(m)} \vdash w^{(m)}\| = \tau \|\tau u^{(m)} - w^{(m)}\| = \tau \|\tau u_1^{(m)} \alpha_1 + \tau z^{(m)} - w^{(m)}\| < \tau(\|\alpha_1\| + r)$$

where we have used that $\|u_1^{(m)}\| < \frac{1}{\tau}$ for $m \in \mathbb{N}$ sufficiently large. We have constructed an infinite sequence of elements of Λ in a bounded region, therefore Λ cannot be uniformly discrete, hence it is not Delone.

In (1) and (2) of this proof, we have constructed the affine mapping ϕ and a $\mathbb{Z}[\tau]$-module L such that $\phi(\Lambda) \subset L$. Take M to be the $\mathbb{Z}[\tau]$-submodule of L generated by $\phi(\Lambda)$. M has a basis, say α_i, $i = 1, \ldots, n$. The star map α_i^* is well determined by the semilinearity from $\beta_i^* = \beta_i$. The vectors α_i^* form a basis of the $\mathbb{Z}[\tau]$ module M^*. To show that a basis of M is contained in $\phi(\Lambda)$, it suffices to show that its star map belongs to $\phi^*(\Lambda)$. Without loss of generality, suppose that $0 \in \langle\phi^*(\Lambda)\rangle^\circ$. Otherwise we take a point $y_0^* \in \langle\phi^*(\Lambda)\rangle^\circ$ and redefine ϕ to ψ given by $\psi(x) = \phi(x) - y_0$. Due to lemma 4.6 we obtain

$$M^* \cap \langle\phi^*(\Lambda)\rangle^\circ \subset \phi^*(\Lambda). \tag{25}$$

Clearly, since $0 \in \langle\phi^*(\Lambda)\rangle^\circ$, we can multiply the basis α_i^* by a suitable τ^k small enough, such that the resulting vectors would belong to $\langle\phi^*(\Lambda)\rangle^\circ$ and retain the property of a basis. □

The following corollary generalizes corollary 3.7 to dimensions greater than one. It is a consequence of the proof of lemma 5.1.

Corollary 5.2. Let $S \subset \mathbb{R}^n$ be a finite set containing the origin. The τ-inflation closure $S^{\vdash}$ is a Delone set if and only if there exists linearly independent vectors $u_1, \ldots, u_n \in S$, such that for any point $x \in S$, one has

$$x = \sum_{i=1}^{n} x_i u_i \qquad x_i \in \mathbb{Q}[\tau].$$

The following lemma is used in the proof of the main theorem. However, it is of an independent interest. It restates the well known [9] implication that boundedness of Ω in a cut and project scheme assures the Delone property of the resulting set, completing it into a necessary and sufficient condition.

Lemma 5.3. Let $\Omega \subset \mathbb{R}^n$ be a convex region. Then $\Sigma(\Omega)$ is Delone if and only if Ω is bounded and it is not embedable into any $(n-1)$-dimensional linear manifold.

Proof. We show only the implication not proven in [9], namely if Ω is convex and $\Sigma(\Omega)$ a Delone set, then Ω is bounded and it cannot be embedded into a linear manifold in $\mathbb{R}^n$ of dimension $n-1$.

If Ω is embeddable into a linear manifold of dimension $k < n$, so does the quasicrystal set $\Sigma(\Omega)$. Therefore it is not relatively dense in $\mathbb{R}^n$ and hence not Delone.

Suppose now that Ω is not embedable into a manifold of lower dimension and that it is not bounded. To show that $\Sigma(\Omega)$ is not uniformly discrete, it is convenient to use [6, theorem 5.7]. It says that the upper bound for the minimal distance between quasicrystal points for $\Sigma(\Omega)$, with convex centrally symmetric acceptance window Ω, is given by

$$\varepsilon(\Omega) \leqslant \frac{4(2\tau-1)}{\sqrt{\pi}} \cdot \sqrt[n]{\frac{\Gamma(n/2+1)|\det\alpha||\det\alpha^*|}{\mathrm{vol}(\Omega)}} \tag{26}$$

where Γ is the gamma function. The larger is the volume of Ω, the smaller are the distances. Minimal distance in a subset of $\Sigma(\Omega)$ is the upper bound for the minimal distances in $\Sigma(\Omega)$, therefore it suffices to find a sequence of subsets of Ω, satisfying assumptions for (26) with growing volume. Let us proceed in two steps.

First we show that any simplex T in $\mathbb{R}^n$ of volume v contains a centrally symmetric convex subset of volume $2^{(1-n)n/2}v$. In T we first find a simplex S as the convex hull of a vertex P of T, and the centres of n edges of T, which meet in P. Denote by C the centre of the centrally symmetric convex subset Q of the $(n-1)$-dimensional face of S opposite to P, which is found by induction. The desired centrally symmetric convex subset of T of volume $2^{1-n} \cdot 2^{(2-n)(n-1)/2}v = 2^{(1-n)n/2}v$ is obtained as the smallest centrally symmetric (with centre C) convex set containing the face Q and the point P. The first step of induction (two-dimensional case) is illustrated in figure 3.

Secondly we show that any convex unbounded set contains a simplex of volume larger than arbitrarily chosen constant. Without loss of generality suppose that $0 \in \Omega$. Since Ω spans $\mathbb{R}^n$, and contains zero, there exists linearly independent $x_1, \ldots, x_n \in \Omega$. The set

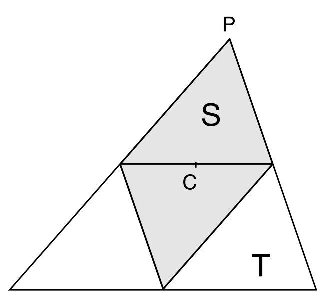

Figure 3. In any triangle T of volume v, one finds a convex centrally symmetric subset (grey coloured) of volume $\frac{1}{2}v$. The notation corresponds to the proof of lemma 5.3.

$\langle 0, x_1, \ldots, x_n \rangle$ is a simplex. Let $r > 0$ be such that $\langle 0, x_1, \ldots, x_n \rangle$ is contained in the ball $B(0, r)$, centred at origin of radius r.

For a direction $k \in \mathbb{R}^n$ of unit norm, we denote the linear manifolds orthogonal with respect to k, by

$$H_{k,a} := \{x \in \mathbb{R}^n \mid (x \mid k) = a\}$$

and the maximal cross section parallel to $H_{k,a}$ by

$$\mathrm{sect}(\Omega, k) := \sup\{\mathrm{vol}(\Omega \cap H_{k,a}) \mid a \in \mathbb{R}\}.$$

The minimal among them is

$$q := \inf\{\mathrm{sect}(\Omega, k) \mid k \in \mathbb{R}^n\} > 0.$$

Since Ω is unbounded, for any $\alpha > 0$ there exists a point $x \in \Omega$ such that $\|x\| > \alpha + r$. There is

$$\mathrm{sect}(\Omega, k) \geqslant q \qquad \text{for } k = \frac{x}{\|x\|}.$$

Let $a \in \mathbb{R}$ be the argument for which $\mathrm{vol}(\Omega \cap H_{k,a}) = \mathrm{sect}(\Omega, k)$. The volume of the simplex T, determined by the face $\Omega \cap H_{k,a}$ and by the point x, satisfies

$$\mathrm{vol}(T) > \frac{1}{n}\,\mathrm{sect}(\Omega, k) \cdot \alpha > \frac{1}{n} q\alpha.$$

For any positive constant δ there exists centrally symmetric convex subset $\widetilde{\Omega}$ of Ω such that $\varepsilon(\widetilde{\Omega}) < \delta$, therefore $\Sigma(\Omega)$ is not uniformly discrete, hence also not Delone. □

Due to the previous preparations, the proof of theorem 2.4 is straightforward.

Proof of theorem 2.4. Due to lemma 5.1, there exists a linear mapping ϕ and a $\mathbb{Z}[\tau]$-module $M := \sum \mathbb{Z}[\tau]\alpha_i$ such that $0, \alpha_i \in \phi(\Lambda) \subset M$. We define the star map by $x = \sum_{i=1}^n x_i \alpha_i \to x^* = \sum_{i=1}^n x_i' \alpha_i$. Since the origin and the vectors of the basis are contained in $\phi(\Lambda)$ and $\alpha_i^* = \alpha_i$, we can use lemma 4.4 to obtain $\phi^*(\Lambda) \supset \{0, \alpha_1, \ldots, \alpha_n\}^{\dashv} = M^* \cap \langle 0, \alpha_1, \ldots, \alpha_n \rangle$. There is $\langle 0, \alpha_1, \ldots, \alpha_n \rangle \subset \langle \phi^*(\Lambda) \rangle$. According to lemma 4.3 we have $\langle \phi^*(\Lambda) \rangle^\circ \cap M^* \subseteq \phi^*(\Lambda)$. Now denote $\Omega := \langle \phi^*(\Lambda) \rangle^\circ \cup \phi^*(\Lambda)$. Clearly, $\phi(\Lambda) = \Sigma(\Omega)$, where Ω° is convex. Since Ω has a nonempty interior ($\Omega^\circ \supset \langle 0, \alpha_1, \ldots, \alpha_n \rangle^\circ$), it is not embedable into a linear manifold of lower dimension. Therefore one can use lemma 5.3 to conclude that Ω is bounded. □

In the proof of theorem 2.4 we find Ω as the union of the convex part $\langle \phi^*(\Lambda) \rangle^\circ$ with a set of points in $\phi^*(\Lambda)$ found on the boundary of $\langle \phi^*(\Lambda) \rangle$. In general, the boundary of $\langle \phi^*(\Lambda) \rangle$ may contain points of $M^* \setminus \phi^*(\Lambda)$. Precisely these points prevent the equality in the following:

$$\Sigma\left(\langle \phi^*(\Lambda) \rangle\right) \neq \Sigma(\Omega) = \phi(\Lambda).$$

The fact that Ω° is convex, does not describe Ω completely. Therefore one needs more than theorem 2.4 alone, to formulate a necessary and sufficient condition for $\Sigma(\Omega)$ to be closed under quasiaddition. For that we introduce the notion of quasiconvexity.

Definition 5.4. Let M be a $\mathbb{Z}[\tau]$-module in $\mathbb{R}^n$. A set $\Omega \subset \mathbb{R}^n$ is *quasiconvex* iff for any linear manifold $W \subset \mathbb{R}^n$ of dimension $0 < k \leqslant n$, Ω satisfies

$$\langle \Omega \cap W \rangle^\circ \cap [\Omega \cap W \cap M^*]^\tau \subset \Omega$$

where $[\Omega \cap W \cap M^*]^\tau$ is defined by (1), and the interior in $\langle \Omega \cap W \rangle^\circ$ is meant with respect to the manifold W.

The complete correspondence between cut and project and τ-inflation invariant sets is given in the proposition below, which follows from lemma 4.6.

Proposition 5.5. Let M be a $\mathbb{Z}[\tau]$-module in $\mathbb{R}^n$. The set $\Sigma(\Omega)$ is closed under quasiaddition if and only if its acceptance window Ω is quasiconvex.

Finally, let us consider an example of a quasicrystal with quasiconvex acceptance window in the $M_2 \subset \mathbb{R}^2$, the $\mathbb{Z}[\tau]$-span of the root system Δ_2 of the noncrystallographic Coxeter group H_2. A basis α_i, $i = 1, 2$, of such a module is often modelled by the fifth roots of unity in the complex plane,

$$\alpha_1 = \alpha_1^* = 1 \qquad \alpha_2 = \mathrm{e}^{4\mathrm{i}\pi/5} \qquad \alpha_2^* = \mathrm{e}^{8\mathrm{i}\pi/5}. \tag{27}$$

Consider $\Omega \subset \mathbb{R}^2$ to be the union of following three sets:

$$\left\{(x, y) \in \mathbb{R}^2 \mid x = t\cos\phi,\ y = t\sin\phi,\ \phi \in \left[0, \frac{2\pi}{5}\right), 0 \leqslant t < 2\right\}$$
$$\left\{(x, y) \in \mathbb{R}^2 \mid x = 2\cos\phi,\ y = 2\sin\phi,\ \phi \in \left[0, \frac{\pi}{5}\right]\right\}$$
$$\left\{(x, y) \in \mathbb{R}^2 \mid x = t\cos\frac{2\pi}{5},\ y = t\sin\frac{2\pi}{5},\ t \in [0, 2) \cap 2\mathbb{Z}[\tau]\right\}.$$

Clearly, this set has the convex interior (disc segment), however is not convex (the line segment forming a part of boundary). Nevertheless, $\Sigma(\Omega)$ is closed under τ-inflation.

6. Example: Inflation closures of three points as quasicrystals

Let us apply the result of this paper to an example of a finitely generated set. Consider the inflation closure $S^{\vdash}$ of three points in the plane. Without loss of generality, one of them can be taken to be the origin, $S = \{0, x_1, x_2\}$. The set $S^{\vdash}$ is invariant under quasiaddition, therefore according to the result of this paper it can be affinely mapped into a cut and project quasicrystal. Whatever are the vectors x_1, x_2, the resulting $S^{\vdash}$ is unique, up to an affine mapping.

Let us consider two 3-point sets in the complex plane for our example,

$$\begin{aligned} S_1 &= \left\{0, 1, \mathrm{e}^{\frac{3\pi\mathrm{i}}{5}}\right\} \\ S_2 &= \left\{0, 1, \mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}\right\}. \end{aligned} \tag{28}$$

According to the main theorem of this paper (theorem 2.4), the inflation closures of S_1 and S_2 can be identified with cut and project quasicrystals.

The example was chosen in such a way that one can find the affine mapping for both sets $S_1^{\vdash}$, $S_2^{\vdash}$ simultaneously. The stage for our quasicrystals will be the $\mathbb{Z}[\tau]$-lattice $M_2 = \sum_i \mathbb{Z}[\tau]\alpha_i$, based on simple roots of the noncrystallographic Coxeter group H_2 as the basis α_1, α_2. We determine the affine mapping $\phi : \mathbb{C} \to M_2$, by setting

$$\phi(0) = 0 \qquad \phi(1) = \alpha_1 \qquad \phi(\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}) = \alpha_2.$$

The most simple definition of a star map on a $\mathbb{Z}[\tau]$-lattice, is to put $\alpha_i^* = \alpha_i$. Since we are using a well known module M_2, we shall consider the standard star map [4], which is determined by

$$\alpha_1^* = \alpha_1 \qquad \alpha_2^* = -\alpha_1 - \tau\alpha_2$$

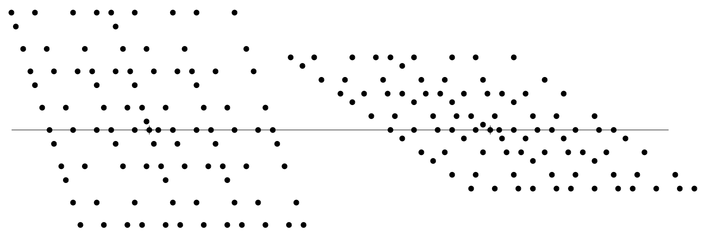

Figure 4. Fragments of quasicrystals (30). The parallelogram shapes of the fragments are chosen in order to facilitate the identification of points on the two pictures according to the mapping ψ (29).

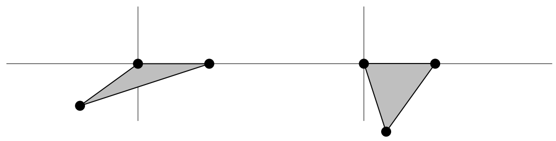

Figure 5. Acceptance windows of the quasicrystals $S_1^{\vdash}$, $S_2^{\vdash}$ from figure 4. The triangles are the convex hulls $\langle S_1^* \rangle$, $\langle S_2^* \rangle$, respectively.

(cf (27)). The latter maps the root system of H_2 onto itself, so that α_2 is also one of the roots.

The embedding of sets S_1, S_2 into M_2 is given by

$$\phi(S_1) = \{0, \alpha_1, \alpha_1 + \tau\alpha_2\}$$
$$\phi(S_2) = \{0, \alpha_1, \alpha_2\}.$$

The pairs of vectors $\{\alpha_1, \alpha_1 + \tau\alpha_2\}$ and $\{\alpha_1, \alpha_2\}$ are related by the matrix

$$\psi = \begin{pmatrix} 1 & \tau' \\ 0 & -\tau' \end{pmatrix} \tag{29}$$

with determinant $\det\psi = -\tau' = \tau^{-1}$, therefore both pairs are bases of the same $\mathbb{Z}[\tau]$-module M_2.

Having embedded the sets $S_i^{\vdash}$ into a $\mathbb{Z}[\tau]$-lattice, which is equipped by a star map, we can identify the sets with cut and project quasicrystals, which means that we can find the corresponding acceptance windows. Since both sets $\phi(S_i)$ are formed by the origin and basis vectors of the $\mathbb{Z}[\tau]$-lattice M_2, we can use corollary 4.5, to conclude that the corresponding acceptance windows are given by the convex hulls of $\phi(S_1^*)$, $\phi(S_2^*)$ respectively,

$$S_i^{\vdash} = \Sigma(\langle S_i^* \rangle) \qquad i = 1, 2. \tag{30}$$

The corresponding acceptance triangles are displayed on figure 5.

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